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ORIGINAL PAPER

Indecomposable representations and boson realizations of the nonlinear deformed angular momentum algebra of Witten's first type

YouNing Li · HuaJun Huang · Dong Ruan

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Abstract In this paper indecomposable representations and boson realizations of the nonlinear angular momentum algebra $\mathcal{R}_{q,p}^{c_1,c_2,c_3}$ of Witten's first type are investigated in a purely algebraic manner. Explicit form of the master representation of $\mathcal{R}_{q,p}^{c_1,c_2,c_3}$ on the space of its universal enveloping algebra is given. Then, from this master representation, other indecomposable representations are obtained in explicit form. Various kinds of single-boson, single inverse boson, and double-boson realizations of $\mathcal{R}_{q,p}^{c_1,c_2,c_3}$ are respectively obtained by generalizing the Holstein–Primakoff realization, the Dyson realization, and the Jordan–Schwinger realization of the Lie algebras SU(2) and SU(1,1). For each kind, the unitary realization, the nonunitary realization, and their connection by the corresponding similarity transformation are respectively discussed. Using a kind of double-boson realizations, the irreducible representation of $\mathcal{R}_{q,p}^{c_1,c_2,c_3}$ in the angular momentum basis is given.

Keywords Deformed algebra, Indecomposable representation, Irreducible representation, Boson realization, Inversion boson realization

Mathematics Subject Classification (2000) 81R

1 Introduction

Nonlinear algebras refer to some specific deformations of the usual algebras obtained by introducing deformation parameters, to which they reduce in the limiting case in which the deformation parameters are set equal to unity. In 1982, Kulish [1] showed

Y. Li · H. Huang · D. Ruan (✉)
Department of Physics, Tsinghua University, Beijing 100084, People's Republic of China
e-mail: dongruan@tsinghua.edu.cn

that the algebra that governs the XXZ-Heisenberg spin model was a deformation of the Lie algebra SU(2), called nowadays $\mathrm{SU}_q(2)$. Since then, the development of quantum groups (Quantized universal enveloping algebras, also called q-algebras or quantum algebras) [2,3] motivated great interest in various deformations of algebraic structures. There are many works devoted to various types of nonlinear algebras due to their interesting mathematical structure [4–9] and possible applications to several research areas of physics including field theory [10–13], statistical mechanics [14–18], nuclear physics [19–21]. In molecular physics [22–28], some simple quantum groups $\mathrm{SU}_q(2)$, $\mathrm{SU}_q(1,1)$, and $\mathrm{SU}_{pq}(2)$ have been extensively applied in describing physical models of diatomic molecules and polyatomic molecules, such as vibrational spectra, rotational spectra, molecular backbending (bandcrossing), and q-deformed vibron model, and discussing, plus a rigid rotator, quasi-molecular resonances in the systems $^{12}\mathrm{C} + {}^{12}\mathrm{C}$ and $^{12}\mathrm{C} + {}^{16}\mathrm{O}$.

The nonlinear Lie algebra to be discussed in this article is a kind of multi-parametric deformed algebras, which can be viewed as the generalization of the quadratic deformation of SU(2) put forward by Witten [11] in his studying Jone's polynomials in node theories and connections with the vertex models in two-dimensional statistical mechanics. Here we denote it by $\mathcal{R}_{q,p}^{c_1,\ldots,c_s}$, whose three elements J_μ $(\mu = 3, -, +)$ satisfy the following commutation relations

$$\begin{aligned} [J_3,\ J_-]_q &= -pJ_-, \\ [J_3,\ J_+]_{\frac{1}{q}} &= \frac{p}{q}J_+, \\ [J_+,\ J_-] &= P(J_3) - P(qJ_3 - p), \end{aligned} \tag{1}$$

where q and p are real numbers, $[X,\ Y]_q \equiv XY - qYX$ is a q-deformed commutator, and $P(J_3)$ is a polynomial function of the Cartan element J_3, i.e.,

$$P(J_3) = \sum_{i=1}^{s} C_i J_3^i, \tag{2}$$

where coefficients C_i are all real numbers, so that, for $q \neq 1$ or $C_s \neq 0$, the highest degree of the power series of J_3 that $[J_+,\ J_-]$ produces is s.

For $\mathcal{R}_{q,p}^{c_1,\ldots,c_s}$, there does exist a Casimir invariant of the type considered by Polychronakos [4] and Roček [5] as follows

$$\mathcal{C} = J_+J_- + P(qJ_3 - p) = J_-J_+ + P(J_3). \tag{3}$$

It is easy to check that $\mathcal{C}$ commutes with all three elements J_μ, i.e., $[\mathcal{C},\ J_\mu] = 0$.

Different from the quantum group $\mathrm{SU}_q(2)$ [29] and the polynomial angular momentum algebra (PAMA) [6,9], both the deformed commutators and the power series of J_3 appear in the algebraic structure (1) of $\mathcal{R}_{q,p}^{c_1,\ldots,c_s}$. That is to say, $\mathcal{R}_{q,p}^{c_1,\ldots,c_s}$ includes $\mathrm{SU}_q(2)$ and PAMA as its special cases. It is obvious that when $q = p = 1$, $\mathcal{R}_{1,1}^{c_1,\ldots,c_s}$ becomes the PAMA with the highest degree of J_3 being $s - 1$ rather than s. When $q = p = 1,\ C_1 = C_2 = 1$ (or $C_1 = C_2 = -1$), and $C_i = 0\,(i = 3, 4, \ldots s)$, $\mathcal{R}_{1,1}^{1,1,\dot{0}}$

(or $\mathcal{R}_{1,1}^{-1,-1,\dot{0}}$, where $\dot{0}$ implies that the number 0 is repeated as many times as necessary) turns to be the usual angular momentum algebra SU(2) (or its non-compact type SU(1,1)) [30]. If let $q = p = 1$, $C_1 = 1$, and $C_i = 0\,(i = 2, 3, \ldots s)$, then $\mathcal{R}_{1,1}^{1,\dot{0}}$ becomes the standard Heisenberg–Weyl algebra [31]. Hence, $\mathcal{R}_{q,p}^{c_1,\ldots,c_s}$ can be viewed as a type of $(s+2)$-parameteric deformations of SU(2) (or SU(1,1)) or Heisenberg–Weyl algebra.

Indecomposable representations (i.e., reducible but not completely reducible representations) of Lie algebras have been found useful in physics for a long time [32–36]. For example, in the quantum mechanical systems, the indecomposable representations have been applied to quasi-exactly soluble potentials [37]. Boson realizations of Lie algebras have played a central role in the study of algebraic models for atomic and molecular structures [38,39]. For a general knowledge of the various types of boson realizations and their applications in physics, one can refer to Klein's review article [40].

In this paper, we shall study in detail indecomposable representations and boson realizations of the five-parametric deformed algebra $\mathcal{R}_{q,p}^{c_1,c_2,c_3} \equiv \mathcal{R}_{q,p}^{c_1,c_2,c_3,\dot{0}}$. First, following the idea presented by Jacobson [43] and Dixmier [44], we shall use the purely algebraic method [41] to calculate the master representation of $\mathcal{R}_{q,p}^{c_1,c_2,c_3}$ on its universal enveloping algebra $U(\mathcal{R})$. This master representation may induce the corresponding representations on quotient spaces $U(\mathcal{R})/I_i$ defined by different left ideals I_is with respect to $U(\mathcal{R})$. It is well known that for the ordinary Lie algebras and PAMAs, their inhomogenous boson realizations can be directly obtained by mapping the corresponding indecomposable representations into the Fock representations [9,42], because these algebras have the same ordinary commutator (Lie product) as the Heisenberg–Weyl algebras generated by the needed sets of boson operators. However, for this kind of multi-parametric deformed algebra $\mathcal{R}_{q,p}^{c_1,\ldots,c_s}$, there exists no simple mapping relation between its indecomposable representations and the Fock representations, so we can not obtain the corresponding boson realizations of $\mathcal{R}_{q,p}^{c_1,c_2,c_3}$. Here, we shall apply an alternative method to studying various boson realizations of $\mathcal{R}_{q,p}^{c_1,c_2,c_3}$, which are analogous to the well-known results of SU(2) and SU(1,1) [30,40].

This paper is arranged as follows. In Sect. 2, the master representation of $\mathcal{R}_{q,p}^{c_1,c_2,c_3}$ is calculated, then from it, various indecomposable representations of $\mathcal{R}_{q,p}^{c_1,c_2,c_3}$ on different quotient spaces will be discussed. In Sect. 3, the single-boson realization and two kinds of double-boson realizations of $\mathcal{R}_{q,p}^{c_1,c_2,c_3}$ are respectively studied in detail, which includes unitary realizations, non-unitary realizations, and their connections. Then, the irreducible representation of $\mathcal{R}_{q,p}^{c_1,c_2,c_3}$ is calculated by using one of double-boson realizations. A simple discussion is given in the final section.

In the following $\mathbb{N}$ denotes the set of positive integers and $\mathbb{C}$ the set of complex numbers.

2 Indecomposable representations of $\mathcal{R}_{q,p}^{c_1,c_2,c_3}$

Accoring to the definition of Eq. (1), the commutation relations complied by $\mathcal{R}_{q,p}^{c_1,c_2,c_3}$ read

$$\begin{aligned}[J_3, J_-]_q &= -pJ_-, \\ [J_3, J_+]_{\frac{1}{q}} &= \frac{p}{q} J_+, \\ [J_+, J_-] &= p\left(C_1 - C_2 p + C_3 p^2\right) + \left(C_1 - C_1 q + 2C_2 pq - 3C_3 p^2 q\right) J_3 \\ &\quad + \left(C_2 - C_2 q^2 + 3C_3 pq^2\right) J_3^2 + \left(C_3 - C_3 q^3\right) J_3^3.\end{aligned} \tag{4}$$

By means of the Poincaré–Birkhoff–Witt theorem [43,44], the basis for $U(\mathcal{R})$ can be expressed as

$$\{X_{(n,m,r)} = J_+^n J_-^m J_3^r | n, m, r \in \mathbb{N}\}, \tag{5}$$

where the identity operator is obtained by setting n, m, r to be zero simultaneously, i.e., $\mathbf{1} = X_{(0,0,0)}$. By acting with the generators of $\mathcal{R}_{q,p}^{c_1,c_2,c_3}$ from the left on the basis (5), we obtain

$$\begin{aligned}\rho(J_3)X_{(n,m,r)} &= q^{m-n} X_{(n,m,r+1)} - p\frac{1-q^{m-n}}{1-q} X_{(n,m,r)}, \\ \rho(J_+)X_{(n,m,r)} &= X_{(n+1,m,r)}, \\ \rho(J_-)X_{(n,m,r)} &= X_{(n,m+1,r)} + C_3 Q_m(3,-3,0) X_{(n-1,m,r+3)} \\ &\quad + \{3C_3 p[Q_m(3,-3,1) - Q_m(2,-2,1)] + C_2 Q_m(2,-2,0)\} X_{(n-1,m,r+2)} \\ &\quad + \Big\{3C_3 p^2 [Q_m(3,-3,2) - 2Q_m(2,-2,2) + Q_m(1,-1,2)] \\ &\quad +2C_2 p [Q_m(2,-2,1) - Q_m(1,-1,1)] + C_1 Q_m(1,-1,0)\Big\} X_{(n-1,m,r+1)} \\ &\quad +p \Big\{C_3 p^2 [Q_m(3,-3,3) - Q_m(2,-2,3) + Q_m(1,-1,3)] \\ &\quad +C_2 p [Q_m(2,-2,2) - Q_m(1,-1,2)] + C_1 Q_m(1,-1,1)\Big\} X_{(n-1,m,r)},\end{aligned} \tag{6}$$

where

$$Q_m(x, y, z) \equiv q^{x(m+1)} \frac{1-q^{yn}}{(1-q)^z}, \quad x, y, z \in \mathbb{N}. \tag{7}$$

It is not difficult to verify that the map ρ forms a representation, i.e.,

$$\begin{aligned}[\rho(J_3), \rho(J_-)]_q &= -p\rho(J_-), \\ [\rho(J_3), \rho(J_+)]_{\frac{1}{q}} &= \frac{p}{q}\rho(J_+), \\ [\rho(J_+), \rho(J_-)] &= p(C_1 - C_2 p + C_3 p^2) + \left(C_1 - C_1 q + 2C_2 pq - 3C_3 p^2 q\right) \rho(J_3) \\ &\quad + \left(C_2 - C_2 q^2 + 3C_3 pq^2\right) \rho(J_3)^2 + \left(C_3 - C_3 q^3\right) \rho(J_3)^3.\end{aligned}$$

We call ρ the master representation of $\mathcal{R}_{q,p}^{c_1,c_2,c_3}$ on $U(\mathcal{R})$. Owing to the fact that the values for the indices m and r do not decrease under the action of ρ, the representation ρ given by Eq. (6) is indecomposable in m and r.

From Eq. (6) it is obvious that the operators $\rho(J_3)$ and $\rho(J_\pm)$ can increase and decrease the power n of J_+, while only increase the powers m and r of J_- and J_3. Therefore, for a fixed $M, R \in \mathbb{N}$,

$$V(M, R) = \left\{X_{(n,m+M,r+R)} = J_+^n J_-^{m+M} J_3^{r+R} | n, m, r \in \mathbb{N}\right\} \tag{8}$$

is the subspace of $U(\mathcal{R})$. Note that $V(0, 0) = U(\mathcal{R})$. $V(M, R)$ is invariant under the action of the representation ρ, so we may get the subduced representation of $\mathcal{R}_{q,p}^{c_1,c_2,c_3}$ through the restriction of the representation ρ on the subspace $V(M, R)$. Furthermore, on the quotient space $U(\mathcal{R})/V(M, R)$, the induced representation of $\mathcal{R}_{q,p}^{c_1,c_2,c_3}$ may be obtained from the representation ρ by setting $X_{(n,m+M,r+R)} \to 0$ formally. Next, we shall consider several other quotient spaces $U(\mathcal{R})/I_i$, where I_is denote the left ideals with respect to $U(\mathcal{R})$.

(1) Consider the left ideal I_1 generated by one element $J_3 - \Lambda\mathbf{1}$ ($\Lambda \in \mathbb{C}$), then the corresponding quotient space $U(\mathcal{R})/I_1$ is spanned by

$$\left\{X_{(n,m)} \equiv X_{(n,m,0)} \ \mathrm{mod} I_1 | n, m \in \mathbb{N}\right\}, \tag{9}$$

therefore the representation ρ given by Eq. (6) may induce a representation, denoted by ρ_1, on $U(\mathcal{R})/I_1$, i.e.,

$$\begin{aligned}
\rho_1(J_3)X_{(n,m)} &= \left(q^{m-n}\Lambda - p\frac{1-q^{m-n}}{1-q}\right) X_{(n,m)},\\
\rho_1(J_+)X_{(n,m)} &= X_{(n+1,m)},\\
\rho_1(J_-)X_{(n,m)} &= X_{(n,m+1)} + \Big\{C_3 Q_m(3,-3,0)\Lambda^3\\
&\quad + \{3C_3 p[Q_m(3,-3,1) - Q_m(2,-2,1)] + C_2 Q_m(2,-2,0)\}\Lambda^2\\
&\quad + \Big\{3C_3 p^2 [Q_m(3,-3,2) - 2Q_m(2,-2,2) + Q_m(1,-1,2)]\\
&\quad +2C_2 p [Q_m(2,-2,1) - Q_m(1,-1,1)] + C_1 Q_m(1,-1,0)\Big\}\Lambda\\
&\quad + \Big\{C_3 p^3 [Q_m(3,-3,3) - Q_m(2,-2,3) + Q_m(1,-1,3)]\\
&\quad +C_2 p^2 [Q_m(2,-2,2) - Q_m(1,-1,2)] + C_1 p Q_m(1,-1,1)\Big\}\\
&\quad \times X_{(n-1,m)},
\end{aligned} \tag{10}$$

where the property $\rho_1(J_3)\mathbf{1} = \Lambda\mathbf{1}$ has been utilized. It is obvious that $X(n, m)$ in $U(\mathcal{R})/I_1$ is the eigenvector of the operator $\rho(J_3)$ corresponding to the eigenvalue $q^{m-n}\Lambda - p\frac{1-q^{m-n}}{1-q}$. Notice from Eq. (10) that the values of the index m do not decrease under the action of ρ_1, the representation ρ_1 is indecomposable in m, and has an invariant subspace $V_1(M)$, which is spanned by

$$\left\{X_{(n,m+M)} = J_+^n J_-^{m+M} | n, m \in \mathbb{N}\right\} \tag{11}$$

for a fixed $M \in \mathbb{N}$, and $V_1(0) = U(\mathcal{R})/I_1$. The subduced representation on $V_1(M)$ can be obtained by restricting ρ_1 to the basis (11) while the induced representation of $\mathcal{R}_{q,p}^{c_1,c_2,c_3}$ on the quotient space $U(\mathcal{R})/I_1/V_1(M)$ may be obtained by formally setting $X_{(n,m+M)} \to 0$. When $M = 1$, Eq. (10) gives directly

$$\begin{aligned}
\overline{\rho}_1(J_3)X_{(n)} &= \left(q^{-n}\Lambda - p\frac{1-q^{-n}}{1-q}\right)X_{(n)},\\
\overline{\rho}_1(J_+)X_{(n)} &= X_{(n+1)},\\
\overline{\rho}_1(J_-)X_{(n)} &= \Big\{C_3 Q(3,-3,0)\Lambda^3\\
&\quad + \{3C_3 p[Q(3,-3,1) - Q(2,-2,1)] + C_2 Q(2,-2,0)\}\,\Lambda^2\\
&\quad + \Big\{3C_3 p^2\,[Q(3,-3,2) - 2Q(2,-2,2) + Q(1,-1,2)]\\
&\quad + 2C_2 p\,[Q(2,-2,1) - Q(1,-1,1)] + C_1 Q(1,-1,0)\Big\}\,\Lambda\\
&\quad + p\,\Big\{C_3 p^2\,[Q(3,-3,3) - Q(2,-2,3) + Q(1,-1,3)]\\
&\quad + C_2 p\,[Q(2,-2,2) - Q(1,-1,2)] + C_1 Q(1,-1,1)\Big\}X_{(n-1)},
\end{aligned} \tag{12}$$

where $Q(x, y, z) \equiv Q_0(x, y, z)$. Or equivalently, we may consider the quotient space $U(\mathcal{R})/I_2$ associated with the left ideal I_2, which is generated by two elements $\{J_-,\ J_3 - \Lambda\mathbf{1}\}$ $(\Lambda \in \mathbb{C})$. The basis of $U(\mathcal{R})/I_2$ reads

$$\left\{X_{(n)} \equiv X_{(n,0,0)}\mathrm{mod} I_2 | n \in \mathbb{N}\right\}. \tag{13}$$

Hence, $U(\mathcal{R})/I_1/V_1(M) \sim U(\mathcal{R})/I_2$. If all the coefficients on the right hand side of Eq. (12) are nonzero, then $\overline{\rho}_1$ is irreducible.

The space $V_1(M')$ $(M' \in \mathbb{N})$ with the basis

$$\left\{X_{(n,m+M')} = J_+^n J_-^{m+M'} | n, m \in \mathbb{N}\right\} \tag{14}$$

is an invariant subspace of $V_1(M)$ as long as $M' > M$. Thus, on the quotient space $V_1(M)/V_1(M')$, ρ_1 may also induce a representation. For the special case of $M' = M + 1$, we have

$$\begin{aligned}
\widetilde{\rho}_1(J_3)X_{(n,M)} &= \left(q^{M-n}\Lambda - p\frac{1-q^{M-n}}{1-q}\right)X_{(n,M)},\\
\widetilde{\rho}_1(J_+)X_{(n,M)} &= X_{(n+1,M)},\\
\widetilde{\rho}_1(J_-)X_{(n,M)} &= \Big\{C_3 Q_M(3,-3,0)\Lambda^3\\
&\quad + \{3C_3 p[Q_M(3,-3,1) - Q_M(2,-2,1)] + C_2 Q_M(2,-2,0)\}\,\Lambda^2
\end{aligned} \tag{15}$$

$$
\begin{aligned}
&+\Big\{3C_3p^2\left[Q_M(3,-3,2)-2Q_M(2,-2,2)+Q_M(1,-1,2)\right]\\
&+2C_2p\left[Q_M(2,-2,1)-Q_M(1,-1,1)\right]+C_1Q_M(1,-1,0)\Big\}\,\Lambda\\
&+\Big\{C_3p^3\left[Q_M(3,-3,3)-Q_M(2,-2,3)+Q_M(1,-1,3)\right]\\
&+C_2p^2\left[Q_M(2,-2,2)-Q_M(1,-1,2)\right]+C_1pQ_M(1,-1,1)\Big\}\\
&\times X_{(n-1,M)}.
\end{aligned}
$$

The representation $\widetilde{\rho}_1$ is algebraically equivalent to $\overline{\rho}_1$ given by Eq. (12).

(2) The left ideal I_3 is generated by one element $J_- - \Theta\mathbf{1}$ ($\Theta \in \mathbb{C}$), then, on the quotient space $U(\mathcal{R})/I_3$ spanned by

$$
\left\{X_{(n,r)} \equiv X_{(n,0,r)}\mathrm{mod}I_3 | n, r \in \mathbb{N}\right\}, \tag{16}
$$

the representation ρ given by Eq. (6) induces the following representation

$$
\begin{aligned}
\rho_2(J_3)X_{(n,r)} &= q^{-n}X_{(n,r+1)} - p\frac{1-q^{-n}}{1-q}X_{(n,r)},\\
\rho_2(J_+)X_{(n,r)} &= X_{(n+1,r)},\\
\rho_2(J_-)X_{(n,r)} &= \Theta\sum_{k=0}^{r}\binom{r}{k}p^{r-k}q^kX_{(n,k)} + C_3Q(3,-3,0)X_{(n-1,r+3)}\\
&\quad+\{3C_3p[Q(3,-3,1)-Q(2,-2,1)]+C_2Q(2,-2,0)\}\,X_{(n-1,r+2)}\\
&\quad+\Big\{3C_3p^2\left[Q(3,-3,2)-2Q(2,-2,2)+Q(1,-1,2)\right]\\
&\quad+2C_2p\left[Q(2,-2,1)-Q(1,-1,1)\right]+C_1Q(1,-1,0)\Big\}\,X_{(n-1,r+1)}\\
&\quad+p\Big\{C_3p^2\left[Q(3,-3,3)-Q(2,-2,3)+Q(1,-1,3)\right]\\
&\quad+C_2p\left[Q(2,-2,2)-Q(1,-1,2)\right]+C_1Q(1,-1,1)\Big\}\,X_{(n-1,r)},
\end{aligned}
\tag{17}
$$

where $\binom{r}{k} = \frac{r!}{k!(r-k)!}$ is the usual binomial coefficient, and the property $\rho_2(J_-)\mathbf{1} = \Theta\mathbf{1}$ has been utilized.

If $\Theta = 0$, then the induced representation ρ_2 given by Eq. (17) is indecomposible in r, and has the invariant subspace $V_2(R)$ spanned by

$$
\left\{X_{(n,r+R)} = J_+^nJ_3^{r+R} | n, r \in \mathbb{N}\right\} \tag{18}
$$

for a fixed $R \in \mathbb{N}$. The subspace $V_2(R)$ carries a subrepresentation of $\mathcal{R}_{q,p}^{c_1,c_2,c_3}$, which may be obtained by restricting ρ_2 to the basis (18). Furthermore, on the quotient space $U(\mathcal{R})/I_3/V_2(R)$, ρ_2 induces a representation, which may be obtained by formally setting $X_{(n,r+R)} \to 0$. For the case of $R = 1$, the induced representation on $U(\mathcal{R})/I_3/V_2(1)$ is identical to $\overline{\rho}_1$ given by Eq. (12) with $\Lambda = 0$.

Finally, we discuss another type of representation of $\mathcal{R}_{q,p}^{c_1,c_2,c_3}$ by using the following new basis

$$\left\{J_+^n F(\mathcal{C}, J_3),\ J_-^m G(\mathcal{C}, J_3)|n, m \in \mathbb{N}\right\}, \tag{19}$$

where F and G are the polynomial functions with respect to $\mathcal{C}$ and J_3. The unit element is obtained by $n = 0$ and $F = 1$, $m = 0$ and $G = 1$.

Thus, from Eq. (6), we have

$$\begin{aligned}
\dot\rho(J_3)J_\pm^n &= q^{\mp n} J_\pm^n J_3 - p\frac{1-q^{\mp n}}{1-q} J_\pm^n,\\
\dot\rho(J_\pm)J_\pm^n &= J_\pm^{n+1},\\
\dot\rho(J_-)J_+^n &= J_+^n J_- + C_3 Q(3,-3,0) J_+^{n-1} J_3^3\\
&\quad + \{3C_3 p[Q(3,-3,1) - Q(2,-2,1)] + C_2 Q(2,-2,0)\} J_+^{n-1} J_3^2\\
&\quad + \Big\{3C_3 p^2 [Q(3,-3,2) - 2Q(2,-2,2) + Q(1,-1,2)]\\
&\quad + 2C_2 p (Q(2,-2,1) - Q(1,-1,1)) + C_1 Q(1,-1,0)\} J_+^{n-1} J_3\\
&\quad + \Big\{C_3 p^3 [Q(3,-3,3) - 3Q(2,-2,3) + 3Q(1,-1,3)]\\
&\quad + C_2 p^2 [Q(2,-2,2) - 2Q(1,-1,2)] + C_1 p Q(1,-1,1)\Big\} J_+^{n-1},\\
\dot\rho(J_+)J_-^n &= J_-^n J_+ + C_3 Q(0,3,0) J_-^{n-1} J_3^3\\
&\quad + [3C_3 p Q(0,3,1) - 3C_3 p Q(0,2,1) + C_2 Q(0,2,0)] J_-^{n-1} J_3^2\\
&\quad + \Big\{3C_3 p^2 [Q(0,3,2) - 2Q(0,2,2) + Q(0,1,2)]\\
&\quad + 2C_2 p (Q(0,2,1) - Q(0,1,1)) + C_1 Q(0,1,0)\} J_-^{n-1} J_3\\
&\quad + p\Big\{C_3 p^2 [Q(0,3,3) - 3Q(0,2,3) + 3Q(0,1,3)]\\
&\quad + C_2 p [Q(0,2,2) - 2Q(0,1,2)] + C_1 Q(0,1,1)\} J_-^{n-1}.
\end{aligned} \tag{20}$$

Let us consider the left ideal I_4 generated by two elements $\{\mathcal{C} - \lambda\mathbf{1}, J_3 - \Lambda\mathbf{1}\}$, where $\lambda, \Lambda \in \mathbb{C}$, then on the quotient space $U(\mathcal{R})/I_4$, $\dot\rho$ induces the following representation

$$\begin{aligned}
\dot\rho_1(J_3)J_\pm^n &= \left(q^{\mp n}\Lambda - p\frac{1-q^{\mp n}}{1-q}\right) J_\pm^n,\\
\dot\rho_1(J_\pm)J_\pm^n &= J_\pm^{n+1},\\
\dot\rho_1(J_-)J_+^n &= \Big\{\lambda - C_3 q^{3-3n}\Lambda^3 + \Big\{3C_3 p[Q(3,-3,1) - Q(2,-2,1) + q^2]\\
&\quad -C_2 q^{2-2n}\Big\} \Lambda^2\\
&\quad + \Big\{3C_3 p^2 [Q(3,-3,2) - 2Q(2,-2,2) + Q(1,-1,2) - q]\\
&\quad + 2C_2 p(Q(2,-2,1) - Q(1,-1,1) - q) - C_1 q^{1-n}\Big\} \Lambda\\
&\quad + \Big\{C_3 p^3 [Q(3,-3,3) - 3Q(2,-2,3) + 3Q(1,-1,3) + 1]\\
&\quad + C_2 p^2 [Q(2,-2,2) - 2Q(1,-1,2) - 1]
\end{aligned} \tag{21}$$

$$
\begin{aligned}
&+ C_1[Q(1,-1,1)-1]-C_0\}\Big\} J_+^{n-1},\\
\dot{\rho}_1(J_+)J_-^n = &\Big\{\lambda - C_3q^{3n}\Lambda^3 + \Big\{3C_3p[Q(0,3,1)-Q(0,2,1)] - C_2q^{2n}\Big\}\Lambda^2\\
&+\Big\{3C_3p^2[Q(0,3,2)-2Q(0,2,2)+Q(0,1,2)]\\
&+2C_2p(Q(0,2,1)-Q(0,1,1)) - C_1q^n\Big\}\Lambda\\
&+\Big\{C_3p^3[Q(0,3,3)-3Q(0,2,3)+3Q(0,1,3)]\\
&+C_2p^2[Q(0,2,2)-2Q(0,1,2)]+C_1Q(0,1,1)-C_0\Big\}\Big\} J_-^{n-1},
\end{aligned}
$$

where the properties $\dot{\rho}_1(\mathcal{C})\mathbf{1} = \lambda\mathbf{1}$ and $\dot{\rho}_1(J_3)\mathbf{1} = \Lambda\mathbf{1}$ have been used. It is obvious from Eq. (21) that $J_\pm^n$ in $U(\mathcal{R})/I_4$ are the eigenvectors of $\dot{\rho}_1(J_3)$ corresponding to the eigenvalues $q^{\mp n}\Lambda - p\frac{1-q^{\mp n}}{1-q}$, respectively. This representation is irreducible if all the coefficients in the last two equations are nonzero.

3 Boson realizations of $\mathcal{R}_{q,p}^{c_1,c_2,c_3}$

In this section, we will study various boson realizations of $\mathcal{R}_{q,p}^{c_1,c_2,c_3}$ in detail. Denote t pairs of mutually commuting boson operators by $\{a_i,\ a_i^+ | i = 1,\ 2, \ldots,\ t\}$ (the annihilation operators a_i are adjoint to the creation operators a_i^+, i.e., $a_i = (a_i^+)^\dagger$, $a_i^+ = (a_i)^\dagger$), which satisfy the commutation relations [30]

$$
\begin{aligned}
[a_i,\ a_j^+] &= \delta_{ij},\\
[\hat{n}_i,\ a_j^+] &= \delta_{ij}a_j^+,\\
[\hat{n}_i,\ a_j] &= -\delta_{ij}a_j,
\end{aligned}
\tag{22}
$$

where $\hat{n}_i \equiv a_i^+a_i$ is the particle number operator of the ith boson. Furthermore, the complete set of basis vectors of Fock space

$$
\mathcal{F}_t = \{|n_1n_2\ldots n_t\rangle | n_1,\ n_2, \ldots,\ n_t = 0,\ 1,\ 2, \ldots\} \tag{23}
$$

may be constructed from the vacuum state $|00\ldots0\rangle$ by using the definition

$$
|n_1n_2\ldots n_t\rangle = \frac{(a_1^+)^{n_1}(a_2^+)^{n_2}\ldots(a_t^+)^{n_t}}{\sqrt{n_1!n_2!\ldots n_t!}}|00\ldots0\rangle. \tag{24}
$$

In fact, these basis vectors are the common normalized eigenvectors of $\hat{n}_i$ belonging to eigenvalues n_i respectively, i.e.,

$$
\hat{n}_i|n_1\ldots n_i\ldots n_t\rangle = n_i|n_1\ldots n_i\ldots n_t\rangle, \tag{25}
$$

and satisfy

$$a_i|n_1\ldots n_i\ldots n_t\rangle = \sqrt{n_i}|n_1\ldots n_i-1\ldots n_t\rangle,$$
$$a_i^+|n_1\ldots n_i\ldots n_t\rangle = \sqrt{n_i+1}|n_1\ldots n_i+1\ldots n_t\rangle. \tag{26}$$

Although the boson operators, a_i and a_i^+, do not possess any inverse in a strict sense because of their singular feature, the generalized inverse of these boson operators, denoted by a_i^{-1} and $(a_i^+)^{-1}$ respectively, may be defined by their action on the basis vectors of Fock space [45–47],

$$a_i^{-1}|n_1\ldots n_i\ldots n_t\rangle = \frac{1}{\sqrt{n_i+1}}|n_1\ldots n_i+1\ldots n_t\rangle,$$
$$(a_i^+)^{-1}|n_1\ldots n_i\ldots n_t\rangle = (1-\delta_{n_i,0})\frac{1}{\sqrt{n_i}}|n_1\ldots n_i-1\ldots n_t\rangle. \tag{27}$$

In fact, a_i^{-1} are only the right inverse of a_i since they satisfy

$$a_i a_i^{-1} = I,$$

however, $a_i^{-1}a_i$ are not unity (I) but are given by

$$a_i^{-1}a_i = I - |00\ldots 0\rangle\langle 00\ldots 0|.$$

where $|00\ldots 0\rangle\langle 00\ldots 0|$ is the projection operator on vacuum. Similarly, $(a_i^+)^{-1}$ are only the left inverse of a_i^+. As is seen, a_i^{-1} behave as the creation operators, while $(a_i^+)^{-1}$ as the annihilation operators.

By direct calculations, it is easy to obtain the following commutation relations

$$[(a_i^+)^{-1},\, a_j^+] = \delta_{ij}|00\ldots 0\rangle\langle 00\ldots 0|,$$
$$[\hat{n}_i,\, (a_j^+)^{-1}] = -\delta_{ij}(a_j^+)^{-1}, \tag{28}$$
$$[\hat{n}_i,\, a_j^{-1}] = \delta_{ij}a_j^{-1}.$$

3.1 Single-boson realizations of $\mathcal{R}_{q,p}^{c_1,c_2,c_3}$

The single-boson realizations of $\mathcal{R}_{q,p}^{c_1,c_2,c_3}$ may be chosen in the following form

$$B^{(k)}(J_3) = h^{(k)}(\hat{n}),$$
$$B^{(k)}(J_+) = f^{(k)}(\hat{n})a^k, \tag{29}$$
$$B^{(k)}(J_-) = (a^+)^k g^{(k)}(\hat{n}),$$

where k is an integer, $h^{(k)}(\hat{n})$, $f^{(k)}(\hat{n})$, and $g^{(k)}(\hat{n})$, being real operator functions of $\hat{n}$ only, have to be determined by the commutation relations (4). For $k = 1, 2, 3, \ldots$, we call $B^{(k)}(J_\mu)$ ($\mu = 3, \pm$) the boson realizations of simple type, quadratic type, cubic type, and so on respectively owing to the fact that the action of $B^{(k)}(J_\pm)$ on the basis vector $|n\rangle$ of the Fock space $\mathcal{F}_1$ leads to $|n \mp k\rangle \sim B^{(k)}(J_\pm)|n\rangle$.

(1) For $k > 0$.

Using the first or second equation of Eq. (4), we may obtain the single-variable difference equation satisfied by $h^{(k)}(\hat{n})$

$$h^{(k)}(\hat{n}) - \frac{1}{q} h^{(k)}(\hat{n} + k) = \frac{p}{q}, \tag{30}$$

with the help of the relations

$$\begin{aligned} (a^+)^k f(\hat{n}) &= f(\hat{n} - k)(a^+)^k, \\ a^k f(\hat{n}) &= f(\hat{n} + k) a^k, \quad i = 1, 2. \end{aligned} \tag{31}$$

The solution of Eq. (30) reads

$$h^{(k)}(\hat{n}) = \frac{p}{q - 1} \left(1 - q^{\hat{n}/k - \alpha}\right), \tag{32}$$

where α is an arbitrary real number. It is worth mentioning that $h^{(k)}(\hat{n})$ is independent of the parameters C_i $(i = 1, 2, 3)$.

The third equation of Eq. (4) requires that $f^{(k)}(\hat{n})$ and $g^{(k)}(\hat{n})$ satisfy the following difference equation

$$\begin{aligned} &\left[\prod_{i=1}^{k} (\hat{n} + i)\right] f^{(k)}(\hat{n}) g^{(k)}(\hat{n}) - \left[\prod_{i=1}^{k} (\hat{n} - i + 1)\right] f^{(k)}(\hat{n} - k) g^{(k)}(\hat{n} - k) \\ &= p\left(C_1 - C_2 p + C_3 p^2\right) + \left(C_1 - C_1 q + 2C_2 pq - 3C_3 p^2 q\right) h^{(k)}(\hat{n}) \\ &\quad + \left(C_2 - C_2 q^2 + 3C_3 pq^2\right) \left[h^{(k)}(\hat{n})\right]^2 + C_3 \left(1 - q^3\right) \left[h^{(k)}(\hat{n})\right]^3. \end{aligned} \tag{33}$$

Note that the functions $f^{(k)}(\hat{n})$ and $g^{(k)}(\hat{n})$ do not appear separately but only appear as their product $f^{(k)}(\hat{n}) g^{(k)}(\hat{n})$. Below we will study in more detail the case of $k = 1$.

Inserting Eq. (32) into Eq. (33) and solving it, we obtain

$$\begin{aligned} f^{(1)}(\hat{n}) g^{(1)}(\hat{n}) = \frac{pq^{-3\alpha}(q^{\hat{n}+1} - 1)}{(q - 1)^3 (\hat{n} + 1)} &\Big\{C_1 (q - 1)^2 q^{2\alpha} \\ &+ C_2 p (q - 1) q^\alpha \left(2q^\alpha - q^{\hat{n}+1} - 1\right) \\ &+ C_3 p^2 \left[3q^\alpha \left(q^\alpha - 1\right) + q^{\hat{n}+1} \left(q^{\hat{n}+1} - 3q^\alpha + 1\right) + 1\right]\Big\}. \end{aligned} \tag{34}$$

This solution shows that we may have some freedom in the choice of the functions $f^{(1)}(\hat{n})$ and $g^{(1)}(\hat{n})$:

(a) The unitary boson realization.

The boson realization $B(J_\mu)$ $(\mu = 3, \pm)$ is unitary if they satisfy

$$\begin{aligned} [B(J_3)]^\dagger &= B(J_3), \\ \left[B(J_\pm)\right]^\dagger &= B(J_\mp). \end{aligned} \tag{35}$$

We call Eq. (35) the unitary conditions of the boson realization.

The second equation of Eq. (35) requires that $f^{(1)}(\hat{n}) = g^{(1)}(\hat{n})$, then solving Eq. (34), we may obtain from Eq. (30) a kind of unitary single-boson realization

$$\begin{aligned}
\breve{B}^{(1)}(J_3) &= \frac{p}{q-1}\left(1-q^{\hat{n}-\alpha}\right),\\
\breve{B}^{(1)}(J_+) &= \sqrt{\frac{pq^{-3\alpha}(q^{\hat{n}+1}-1)}{(q-1)^3(\hat{n}+1)}}\left\{C_1(q-1)^2q^{2\alpha}+C_2p(q-1)q^{\alpha}\left(2q^{\alpha}-q^{\hat{n}+1}-1\right)\right.\\
&\quad\left.+C_3p^2\left[3q^{\alpha}(q^{\alpha}-1)+q^{\hat{n}+1}(q^{\hat{n}+1}-3q^{\alpha}+1)+1\right]\right\}^{1/2}a, \qquad (36)\\
\breve{B}^{(1)}(J_-) &= a^+\sqrt{\frac{pq^{-3\alpha}\left(q^{\hat{n}+1}-1\right)}{(q-1)^3(\hat{n}+1)}}\left\{C_1(q-1)^2q^{2\alpha}\right.\\
&\quad+C_2p(q-1)q^{\alpha}\left(2q^{\alpha}-q^{\hat{n}+1}-1\right)\\
&\quad\left.+C_3p^2\left[3q^{\alpha}(q^{\alpha}-1)+q^{\hat{n}+1}(q^{\hat{n}+1}-3q^{\alpha}+1)+1\right]\right\}^{1/2}.
\end{aligned}$$

In order to obtain the real boson realization, the values of n in the matrix elements of $\breve{B}^{(1)}(J_\pm)$ in the Fock space $\mathcal{F}_1 = \{|n\rangle\, | n = 0, 1, 2, \ldots\}$ need limiting for the given $\{q, p, C_1, C_2, C_3\}$. When $q = p = C_1 = C_2 = 1$ and $C_3 = 0$, Eq. (36) becomes the Holstein–Primakoff realization of SU(2) [48]. Hence, we call $\breve{B}^{(k)}(J_\mu)$ the Holstein–Primakoff-like realization of kth order of $\mathcal{R}_{q,p}^{c_1,c_2,c_3}$.

(b) The non-unitary boson realization.

If the unitary conditions (35) need not satisfying, it follows from Eq. (34) that the convenient choice, for example, $g(\hat{n}) = 1$, may immediately give rise to a kind of non-unitary single-boson realization of interest

$$\begin{aligned}
\bar{B}^{(1)}(J_3) &= \frac{p}{q-1}\left(1-q^{\hat{n}-\alpha}\right),\\
\bar{B}^{(1)}(J_+) &= \frac{pq^{-3\alpha}(q^{\hat{n}+1}-1)}{(q-1)^3(\hat{n}+1)}\left\{C_1(q-1)^2q^{2\alpha}\right.\\
&\quad+C_2p(q-1)q^{\alpha}\left(2q^{\alpha}-q^{\hat{n}+1}-1\right)\\
&\quad\left.+C_3p^2\left[3q^{\alpha}(q^{\alpha}-1)+q^{\hat{n}+1}\left(q^{\hat{n}+1}-3q^{\alpha}+1\right)+1\right]\right\}a,\\
\bar{B}^{(1)}(J_-) &= a^+. \qquad (37)
\end{aligned}$$

When $q = p = C_1 = C_2 = 1$ and $C_3 = 0$, Eq. (37) becomes the standard Dyson realization of SU(2) introduced originally by Dyson [49] in his study of spin-wave interactions. Hence, we call $\bar{B}^{(k)}(J_\mu)$ the Dyson-like realization of kth order of $\mathcal{R}_{q,p}^{c_1,c_2,c_3}$. Different from the Holstein–Primakoff-like realization (36), no square-root symbol appears in Eq. (37) so that the Dyson-like realization may not only avoid the convergence questions associated with the expansion of square-root symbol but also make the values of n in $\mathcal{F}_1 = \{|n\rangle\, | n = 0, 1, 2, \ldots\}$ unlimited.

Inserting Eq. (36) or (37) into Eq. (3), we obtain the Casimir operator of $\mathcal{R}_{q,p}^{c_1,c_2,c_3}$

$$\mathcal{C}=\frac{pq^{-3\alpha}(q^{\alpha}-1)}{(q-1)^3}\left\{(q-1)q^{\alpha}\left[C_1(q-1)q^{\alpha}+C_2p(q^{\alpha}-1)\right]+C_3p^2(q^{\alpha}-1)^2\right\}. \tag{38}$$

It is clear that $\mathcal{C}$, which is independent of $\hat{n}$, is a number.

(c) Unitarization of the non-unitary realization.

It is not difficult to find that the non-unitary Dyson-like realization $\bar{B}^{(1)}(J_\mu)$ may be related to the unitary Holstein–Primakoff-like realization $\breve{B}^{(1)}(J_\mu)$ by a similarity transformation S, i.e.,

$$S\bar{B}^{(1)}(J_\mu)S^{-1}=\breve{B}^{(1)}(J_\mu),\quad \mu=3,\ \pm. \tag{39}$$

In general, S is an operator function with respect to $\{a,a^+,n\}$. Using the unitary condition $\breve{B}^{(1)}(J_+)=(\breve{B}^{(1)}(J_-))^\dagger$, we obtain from Eq. (39)

$$U^{-1}\left(\bar{B}^{(1)}(J_-)\right)^\dagger U=\bar{B}^{(1)}(J_+), \tag{40}$$

where $U\equiv S^\dagger S$. Note that $\bar{B}^{(1)}(J_3)$ is already Hermitian, so we call Eq. (40) the unitarization equation of the Dyson-like realization. As an example, let us calculate concretely the explicit expression of S. The first equation of Eq. (39) implies that S commutes with J_3 and is at most the function of $\hat{n}$, thus, calculating the matrix element of Eq. (40) between the basis vectors $\langle n-1|$ and $|n\rangle$ and using Eq. (37), we may derive the equation satisfied by S, i.e.,

$$\begin{aligned}\langle n|S|n\rangle^2=&\frac{pq^{-3\alpha}(q^n-1)}{(q-1)^3n}\Big\{C_1(q-1)^2q^{2\alpha}+C_2p(q-1)q^{\alpha}(2q^{\alpha}-q^n-1)\\&+C_3p^2\left[3q^{\alpha}(q^{\alpha}-1)+q^n(q^n-3q^{\alpha}+1)+1\right]\Big\}\langle n-1|S|n-1\rangle^2.\end{aligned} \tag{41}$$

Solving Eq. (41) with the initial condition $\langle 0|S|0\rangle=\kappa_0$ (κ_0 is a real number) gives

$$\begin{aligned}S(n)=&\kappa_0\left(\frac{p}{(q-1)^3}\right)^{n/2}\prod_{l=0}^{n-1}\frac{1}{\sqrt{l+1}}\Bigg\{C_1\sum_{k=1}^{3}[3(k-1)(k-3)+1]q^{k-\alpha}\left(q^l-q^{-1}\right)\\&+C_2p\sum_{k=0}^{1}(-1)^kq^{k-\alpha}(2-q^{-\alpha}-2q^{l+1}+q^{2i-\alpha+2})\\&+C_3p^2\sum_{k=1}^{3}[(5k-6)(k-3)+1]q^{-k\alpha}\left(q^{k(l+1)}-1\right)\Bigg\}^{1/2}.\end{aligned} \tag{42}$$

Using Eq. (25), $S(n)$ may be written as the operator function $S(\hat{n})$ by replacing $n \to \hat{n}$ in Eq. (42), so that $S(n) = \langle n|S(\hat{n})|n\rangle$.

(2) For $k < 0$.

Using the same method as the case of $k > 0$, we obtain

$$h^{(-1)}(\hat{n}) = \frac{p}{q-1}\left(1 - q^{-\hat{n}-1}\right), \tag{43}$$

and

$$f^{(-1)}(\hat{n})g^{(-1)}(\hat{n}) = -\frac{\hat{n}pq^{-3\hat{n}}(q^{\hat{n}}-1)}{(q-1)^3}\Big\{(q-1)q^{\hat{n}} \times \Big[C_1(q-1)q^{\hat{n}} + C_2p(q^{\hat{n}}-1)\Big] + C_3p^2s(q^{\hat{n}}-1)^2\Big\}, \tag{44}$$

respectively.

Inserting Eqs. (43) and (44) into Eq. (30), we may obtain the explicit expressions for the unitary single inverse boson realization by taking $f^{(-1)}(\hat{n}) = g^{(-1)}(\hat{n})$, i.e.,

$$\begin{aligned}
\breve{B}^{(-1)}(J_3) &= \frac{p}{q-1}\left(1 - q^{-\hat{n}-1}\right),\\
\breve{B}^{(-1)}(J_+) &= \sqrt{-\frac{\hat{n}pq^{-3\hat{n}}(q^{\hat{n}}-1)}{(q-1)^3}}\Big\{(q-1)q^{\hat{n}}\\
&\quad \times \Big[C_1(q-1)q^{\hat{n}} + C_2p(q^{\hat{n}}-1)\Big] + C_3p^2(q^{\hat{n}}-1)^2\Big\}^{1/2} a^{-1},\\
\breve{B}^{(-1)}(J_-) &= (a^+)^{-1}\sqrt{-\frac{\hat{n}pq^{-3\hat{n}}(q^{\hat{n}}-1)}{(q-1)^3}}\Big\{(q-1)q^{\hat{n}}\\
&\quad \times \Big[C_1(q-1)q^{\hat{n}} + C_2p(q^{\hat{n}}-1)\Big] + C_3p^2(q^{\hat{n}}-1)^2\Big\}^{1/2},
\end{aligned} \tag{45}$$

and the non-unitary one by taking $g^{(-1)}(\hat{n}) = 1$, i.e.,

$$\begin{aligned}
\bar{B}^{(-1)}(J_3) &= \frac{p}{q-1}\left(1 - q^{-\hat{n}-1}\right),\\
\bar{B}^{(-1)}(J_+) &= -\frac{\hat{n}pq^{-3\hat{n}}(q^{\hat{n}}-1)}{(q-1)^3}\Big\{(q-1)q^{\hat{n}}\\
&\quad \times \Big[C_1(q-1)q^{\hat{n}} + C_2p(q^{\hat{n}}-1)\Big] + C_3p^2(q^{\hat{n}}-1)^2\Big\} a^{-1},\\
\bar{B}^{(-1)}(J_-) &= (a^+)^{-1}.
\end{aligned} \tag{46}$$

The above two kinds of inverse boson realizations are unfamiliar to us. Now let us see the corresponding results for SU(2). Setting $q = p = C_1 = C_2 = 1$ and $C_3 = 0$ in Eqs. (45) and (46) respectively gives the single inverse boson realizations of SU(2):

$$\begin{aligned}\breve{B}^{(-1)}_{\text{su2}}(J_3) &= \hat{n},\\ \breve{B}^{(-1)}_{\text{su2}}(J_+) &= \hat{n}\sqrt{-(\hat{n}+1)}a^{-1},\\ \breve{B}^{(-1)}_{\text{su2}}(J_-) &= (a^+)^{-1}\hat{n}\sqrt{-(\hat{n}+1)},\end{aligned} \tag{47}$$

and

$$\begin{aligned}\bar{B}^{(-1)}_{\text{su2}}(J_3) &= \hat{n},\\ \bar{B}^{(-1)}_{\text{su2}}(J_+) &= -(\hat{n}+1)\hat{n}^2a^{-1},\\ \bar{B}^{(-1)}_{\text{su2}}(J_-) &= (a^+)^{-1}.\end{aligned} \tag{48}$$

We notice from Eq. (47) that in $\mathcal{F}_1$ the square roots in the matrix elements $\langle n \pm 1| \breve{B}^{(-1)}_{\text{su2}}(J_\pm)|n\rangle$ are either pure imaginary or null.

The similarity transformation $S(\hat{n})$, which transforms the non-unitary realization $\bar{B}^{(-1)}(J_\mu)$ to the unitary realization $\breve{B}^{(-1)}(J_\mu)$, reads

$$\begin{aligned}S(n) = \varepsilon_0 \left(\frac{p}{(q-1)^3}\right)^{-n/2} \prod_{l=0}^{n-1} \sqrt{\frac{-q^{3(l+1)}}{(l+1)(q^{l+1}-1)}} \Big\{C_1(q-1)^3 q^{2(l+1)}\\ +C_2 p(q-1)(q^{l+1}-1)q^{l+1} + C_3p^2(q^{l+1}-1)^2\Big\}^{-1/2},\end{aligned} \tag{49}$$

with the initial condition being $S(0) = \langle 0|S(\hat{n})|0\rangle = \varepsilon_0$ (ε_0 is a real number).

3.2 The first kind of double-boson realizations

Generalizing the famous Jordan–Schwinger realization of SU(2) [30,50], the first kind of double-boson realizations of $\mathcal{R}^{c_1,c_2,c_3}_{q,p}$ may be defined as

$$\begin{aligned}B_1^{(k,l)}(\mathcal{J}_3) &= h_1^{(k,l)}(\hat{n}_1,\hat{n}_2),\\ B_1^{(k,l)}(\mathcal{J}_+) &= f_1^{(k,l)}(\hat{n}_1,\hat{n}_2)(a_1^+)^k a_2^l,\\ B_1^{(k,l)}(\mathcal{J}_-) &= a_1^k(a_2^+)^l g_1^{(k,l)}(\hat{n}_1,\hat{n}_2),\end{aligned} \tag{50}$$

where k and l are integers, $h_1^{(k,l)}(\hat{n}_1,\hat{n}_2)$, $f_1^{(k,l)}(\hat{n}_1,\hat{n}_2)$, and $g_1^{(k,l)}(\hat{n}_1,\hat{n}_2)$ are the operator functions of $\hat{n}_1$ and $\hat{n}_2$, which need determining by the commutation relations (4). For the fixed (k,l), the action of $B_1^{(k,l)}(\mathcal{J}_\pm)$ on some basis vector $|n_1n_2\rangle$ of $\mathcal{F}_2$ gives $|n_1 \pm k, n_2 \mp l\rangle = B_1^{(k,l)}(\mathcal{J}_\pm)|n_1n_2\rangle$.

Using the first or second equation of Eq. (4), we obtain the equation satisfied by $h_1^{(k,l)}(\hat{n}_1,\hat{n}_2)$

$$h_1^{(k,l)}(\hat{n}_1,\hat{n}_2) - \frac{1}{q}h_1^{(k,l)}(\hat{n}_1-k,\hat{n}_2+l) = \frac{p}{q}. \tag{51}$$

Solving Eq. (51) gives

$$h_1^{(k,l)}(\hat{n}_1,\hat{n}_2) = \frac{p}{q-1}\left(1 - q^{-\frac{\hat{n}_1}{2k}+\frac{\hat{n}_2}{2l}}\right). \tag{52}$$

The third equation of Eq. (4) requires that $f_1^{(k,l)}(\hat{n}_1,\hat{n}_2)g_1^{(k,l)}(\hat{n}_1,\hat{n}_2)$ satisfies the following difference equation

$$\begin{aligned}
&\left[\prod_{i=1}^{k}\left(\hat{n}_1 - i + 1\right)\right]\left[\prod_{i=1}^{l}\left(\hat{n}_2 + i\right)\right] f_1^{(k,l)}(\hat{n}_1,\hat{n}_2)g_1^{(k,l)}\left(\hat{n}_1,\hat{n}_2\right)\\
&\quad -\left[\prod_{i=1}^{k}\left(\hat{n}_1 + i\right)\right]\left[\prod_{i=1}^{l}\left(\hat{n}_2 - i + 1\right)\right] f_1^{(k,l)}\left(\hat{n}_1 + k, \hat{n}_2 - l\right) g_1^{(k,l)}\left(\hat{n}_1 + k, \hat{n}_2 - l\right)\\
&\quad = p\left(C_1 - C_2 p + C_3 p^2\right) + \left(C_1 - C_1 q + 2C_2 pq - 3C_3 p^2 q\right) h_1^{(k,l)}(\hat{n}_1,\hat{n}_2)\\
&\quad + \left(C_2 - C_2 q^2 + 3C_3 pq^2\right)\left[h_1^{(k,l)}\left(\hat{n}_1,\hat{n}_2\right)\right]^2 + C_3(1-q^3)\left[h_1^{(k,l)}\left(\hat{n}_1,\hat{n}_2\right)\right]^3.
\end{aligned} \tag{53}$$

However, it is very difficult to obtain the general solutions of Eq. (53) for arbitrary k and l. Below we will study in more detail the simple case of $(k,l) = (1,1)$.

Inserting Eq. (52) into Eq. (53) and solving it, we obtain

$$\begin{aligned}
f_1(\hat{n}_1,\hat{n}_2)g_1(\hat{n}_1,\hat{n}_2) = &\frac{pq^{1-\hat{M}/2}(q^{\hat{n}_1}-1)}{(q-1)^3\hat{n}_1(\hat{n}_2+1)}\Big[-C_1(q-1)^2\\
&+C_2 p(q-1)\left(q^{1+\hat{N}/2} + q^{1-\hat{M}/2} - 2\right)\\
&-C_3 p^2\left(q^{2+\hat{N}} + q^{2-\hat{M}} - 3q^{1+\hat{N}/2} - 3q^{1-\hat{M}/2} + q^{2+\hat{n}_2} + 3\right)\Big],
\end{aligned} \tag{54}$$

here and afterwards, the superscript (1, 1) has been omitted for the sake of simplicity, $\hat{N} = \hat{n}_1 + \hat{n}_2$ is the total particle number operator, and $\hat{M} = \hat{n}_1 - \hat{n}_2$ the particle number difference operator.

In the following, from Eq. (54) we will discuss respectively the unitary realizations and the non-unitary realizations by choosing $f_1(\hat{n}_1,\hat{n}_2)$ and $g_1(\hat{n}_1,\hat{n}_2)$.

(a) The unitary realization.

The unitary conditions (35) now read

$$\begin{aligned}
B_1^\dagger(J_3) &= B_1(J_3),\\
B_1^\dagger(J_\pm) &= B_1(J_\mp).
\end{aligned} \tag{55}$$

Notice that $B_1(J_3)$ is already Hermitian. The two later equations of Eq. (55) require $f_1(\hat{n}_1,\hat{n}_2) = g_1(\hat{n}_1,\hat{n}_2)$. Thus, solving Eq. (54) and substituting the expression of $f_1(\hat{n}_1,\hat{n}_2)$ into Eq. (51), we may obtain

$$\breve{B}_1(J_3) = \frac{p}{q-1}\left(1 - q^{-\hat{M}/2}\right),$$

$$\begin{aligned}\breve{B}_1(J_+) = &\sqrt{\frac{pq^{1-\hat{M}/2}\left(q^{\hat{n}_1}-1\right)}{(q-1)^3\hat{n}_1(\hat{n}_2+1)}}[-C_1(q-1)^2\\ &+C_2 p(q-1)\left(q^{1+\hat{N}/2}+q^{1-\hat{M}/2}-2\right)\\ &-C_3 p^2\left(q^{2+\hat{N}}+q^{2-\hat{M}}-3q^{1+\hat{N}/2}-3q^{1-\hat{M}/2}+q^{2+\hat{n}_2}+3\right)]^{1/2} a_1^+ a_2,\end{aligned} \tag{56}$$

$$\begin{aligned}\breve{B}_1(J_-) = a_1 a_2^+ &\sqrt{\frac{pq^{1-\hat{M}/2}\left(q^{\hat{n}_1}-1\right)}{(q-1)^3\hat{n}_1(\hat{n}_2+1)}}\Big[-C_1(q-1)^2\\ &+C_2 p(q-1)\left(q^{1+\hat{N}/2}+q^{1-\hat{M}/2}-2\right)\\ &-C_3 p^2\left(q^{2+\hat{N}}+q^{2-\hat{M}}-3q^{1+\hat{N}/2}-3q^{1-\hat{M}/2}+q^{2+\hat{n}_2}+3\right)\Big]^{1/2}.\end{aligned}$$

It can be easily checked that the realization (56) satisfies the commutation relations of $\mathcal{R}_{q,p}^{c_1,c_2,c_3}$. We observe that $\breve{B}_1(J_3)$ depends on the particle number difference operator $\hat{M}$ only. Except for the case of $q = p = C_1 = C_2 = 1$ and $C_3 = 0$, Eq. (56) is analogous to the Holstein–Primakoff single-boson realization of SU(2) [48] because of the existence of the square-root symbols. Hence, the acting space of $\breve{B}_1(J_\mu)$ may be certain subspaces of the Fock space $\mathcal{F}_2 = \{|n_1 n_2\rangle \mid n_1, n_2 = 0, 1, 2, \ldots\}$, in which n_1 and n_2 need limiting in order that the values of the square roots appeared in the matrix elements $\langle n_1 \pm 1 n_2 \mp 1|\breve{B}_1(J_\pm)|n_1 n_2\rangle$ must be greater than or equal to zero.

Inserting Eq. (56) into Eq. (3), the Casimir invariant $\mathcal{C}$ of $\mathcal{R}_{q,p}^{c_1,c_2,c_3}$ may be expressed in terms of the boson number operators $\hat{n}_1$ and $\hat{n}_2$ as

$$\mathcal{C} = \frac{\zeta(\hat{N}/2)}{(q-1)^3}\left[-C_1(q-1)^2 + C_2(q-1)\zeta(\hat{N}/2) - C_3\zeta^2(\hat{N}/2)\right], \tag{57}$$

where $\zeta(\hat{N}) = p(q^{1+\hat{N}} - 1)$. Equation (57) shows clearly that $\mathcal{C}$ depends on the total particle number operator $\hat{N}$ only, rather than the particle number difference operator $\hat{M}$ and the separate particle number operator $\hat{n}_1$ or $\hat{n}_2$. Calculating the expectation value $\langle n_1 n_2|\mathcal{C}|n_1 n_2\rangle$, we have

$$\langle n_1 n_2|\mathcal{C}|n_1 n_2\rangle = \frac{\zeta(j)}{(q-1)^3}\left[-C_1(q-1)^2 + C_2(q-1)\zeta(j) - C_3\zeta^2(j)\right], \tag{58}$$

where $j = N/2$ has been introduced, i.e., the values that j may take are half integers ($j = 0, 1/2, 1, \ldots$). The similar conclusion exists for SU(2) [30].

(b) The non-unitary realization.

It follows from Eq. (54) that the choice $g_1(\hat{n}_1, \hat{n}_2) = 1$ may immediately give rise to the non-unitary double-boson realization

$$\begin{aligned}
\bar{B}_1(J_3) &= \frac{p}{q-1}\left(1 - q^{-\hat{M}/2}\right),\\
\bar{B}_1(J_+) &= \frac{pq^{1-\hat{M}/2}(q^{\hat{n}_1}-1)}{(q-1)^3\hat{n}_1(\hat{n}_2+1)}\Big[-C_1(q-1)^2\\
&\quad + C_2 p(q-1)\left(q^{1+\hat{N}/2} + q^{1-\hat{M}/2} - 2\right)\\
&\quad - C_3 p^2\left(q^{2+\hat{N}} + q^{2-\hat{M}} - 3q^{1+\hat{N}/2} - 3q^{1-\hat{M}/2} + q^{2+\hat{n}_2} + 3\right)\Big] a_1^+ a_2,\\
\bar{B}_1(J_-) &= a_1 a_2^+.
\end{aligned} \tag{59}$$

The Casimir invariant $\mathcal{C}$ has the same expression as Eq. (57). Notice that except for the special case of $q = p = C_1 = C_2 = 1$ and $C_3 = 0$, the double-boson realization (59) is in fact analogous to the Dyson single-boson realization of SU(2).

Different from the unitary realization (56), no square-root symbols appear in the nonunitary realization (59), hence, it may not only avoid the convergence questions associated with the expansion of square-root operator but also make the values of n_1 and n_2 in $\{|n_1 n_2\rangle\}$ unlimited, i.e., the acting space of $\bar{B}_1(J_\mu)$ is the whole Fock space $\mathcal{F}_2$.

(c) Unitarization of the non-unitary realization.

Following the same method as used in Sect. 3.1, the non-unitary realization (59) may also be connected with the unitary realization (56) by the corresponding similarity transformation. However, because of the complexity of result (59), here we restrict ourselves to the special case $q = p = 1$, i.e., $\bar{B}'_1(J_\mu) = \lim_{q,p=1} \bar{B}_1(J_\mu)$ and $\breve{B}'_1(J_\mu) = \lim_{q,p=1} \breve{B}_1(J_\mu)$.

Denoting the similarity transformation by S_1, we have

$$S_1 \bar{B}'_1(J_\mu) S_1^{-1} = \breve{B}'_1(\mathcal{J}_\mu), \quad \mu = 3, \pm. \tag{60}$$

The equation for $\mu = 3$ implies that S_1 depends only on the particle number operators, $\hat{n}_1$ and $\hat{n}_2$.

Using Eq. (60) and the unitary conditions $(\breve{B}'_1(J_\pm))^\dagger = \breve{B}'_1(J_\mp)$, we may obtain the following unitarization equations

$$\begin{aligned}
U_1^{-1}\left(\bar{B}'_1(J_3)\right)^\dagger U_1 &= \bar{B}'_1(J_3),\\
U_1^{-1}\left(\bar{B}'_1(J_\pm)\right)^\dagger U_1 &= \bar{B}'_1(J_\mp),
\end{aligned} \tag{61}$$

where $U_1 \equiv S_1^\dagger S_1$ is Hermitian. Calculating the matrix element of the first equation of Eq. (61) in the Fock space $\mathcal{F}_2$, and using Eq. (59) with $q = p = 1$, we may deduce the difference equation satisfied by the expectation value $S_1(n_1, n_2) \equiv \langle n_1 n_2 | S_1 | n_1 n_2 \rangle$,

$$\left\{-4C_1+4C_2(n_2+2)-C_3\left[n_1^2+3(n_2+2)^2\right]\right\}S_1(n_1,n_2)^2$$
$$-4(n_2+1)S_1\left(n_1-1,n_2+1\right)^2=0. \tag{62}$$

Its solution is

$$S_1\left(\hat{n}_1,\hat{n}_2\right)=\sqrt{\frac{C_3^{1-\hat{n}_1}\Gamma(-\hat{n}_2)\Gamma\left(\omega_1^+(\hat{N})\right)\Gamma(\omega_1^-(\hat{N}))}{\Gamma(1-\hat{N})\Gamma\left(\omega_1^+(\hat{N})+\hat{n}_1-1\right)\Gamma\left(\omega_1^-(\hat{N})+\hat{n}_1-1\right)}}, \tag{63}$$

where $\Gamma(\hat{s})$ stands for an operator function of $\hat{s}$, whose expectation value in $\mathcal{F}$ is the ordinary Gamma symbol $\Gamma(s)$ for the integer or real number s, i.e., $\langle n_1n_2|\,\Gamma(\hat{s})\,|n_1n_2\rangle=\Gamma(s)$. The symbol $\omega_1^{\pm}(\hat{x})$ is given by

$$\omega_1^{\pm}(\hat{x})=\frac{1}{4C_3}\left[2C_2+C_3(2-3\hat{x})\ \pm\sqrt{-16C_1C_3+[2C_2-C_3(2+\hat{x})][2C_2+3C_3(2+\hat{x})]}\right].$$

In Eq. (63), the minus sign out of the square-root symbol has been omitted without loss of generality.

3.3 The second kind of double-boson realizations of $\mathcal{R}_{q,p}^{c_1,c_2,c_3}$

The second kind of double-boson realizations of $\mathcal{R}_{q,p}^{c_1,c_2,c_3}$ may be chosen in the form

$$\begin{aligned}B_2^{(k,l)}(J_3)&=h_2^{(k,l)}(\hat{n}_1,\hat{n}_2),\\ B_2^{(k,l)}(J_+)&=f_2^{(k,l)}(\hat{n}_1,\hat{n}_2)(a_1^+)^k(a_2^+)^l,\\ B_2^{(k,l)}(J_-)&=a_1^ka_2^lg_2^{(k,l)}(\hat{n}_1,\hat{n}_2),\end{aligned} \tag{64}$$

where the real operator functions $h_2^{(k,l)}(\hat{n}_1,\hat{n}_2)$, $f_2^{(k,l)}(\hat{n}_1,\hat{n}_2)$, and $g_2^{(k,l)}(\hat{n}_1,\hat{n}_2)$ have to be determined by the commutation relations (4) of $\mathcal{R}_{q,p}^{c_1,c_2,c_3}$.

It follows from the first equation of Eq. (4) that we obtain the equation satisfied by $h_2^{(k,l)}(\hat{n}_1,\hat{n}_2)$

$$h_2^{(k,l)}(\hat{n}_1,\hat{n}_2)-\frac{1}{q}h_2^{(k,l)}(\hat{n}_1-k,\hat{n}_2-l)=\frac{p}{q}. \tag{65}$$

Its solution is given by

$$h_2^{(k,l)}(\hat{n}_1,\hat{n}_2)=\frac{p}{q-1}\left(1-q^{\beta-\frac{\hat{n}_1}{2k}-\frac{\hat{n}_2}{2l}}\right), \tag{66}$$

where β is an arbitrary real number, which will be set as $-1/2$ in order to give the well known results of SU(1,1) by taking $q=p=1$, $C_1=C_2=-1$, and $C_3=0$.

The third equation of Eq. (4) requires that $f_2(\hat{n}_1, \hat{n}_2)g_2(\hat{n}_1, \hat{n}_2)$ satisfies the following equation

$$\left[\prod_{i=1}^{k}\left(\hat{n}_1 - i + 1\right)\right]\left[\prod_{i=1}^{l}\left(\hat{n}_2 - i + 1\right)\right] f_2^{(k,l)}\left(\hat{n}_1, \hat{n}_2\right) g_2^{(k,l)}(\hat{n}_1, \hat{n}_2)$$
$$-\left[\prod_{i=1}^{k}(\hat{n}_1 + i)\right]\left[\prod_{i=1}^{l}(\hat{n}_2 + i)\right] f_2^{(k,l)}\left(\hat{n}_1 + k, \hat{n}_2 + l\right) g_2^{(k,l)}\left(\hat{n}_1 + k, \hat{n}_2 + l\right)$$
$$= p\left(C_1 - C_2 p + C_3 p^2\right) + \left(C_1 - C_1 q + 2C_2 pq - 3C_3 p^2 q\right) h_2^{(k,l)}\left(\hat{n}_1, \hat{n}_2\right)$$
$$+\left(C_2 - C_2 q^2 + 3C_3 pq^2\right)\left[h_2^{(k,l)}\left(\hat{n}_1, \hat{n}_2\right)\right]^2 + C_3(1 - q^3)\left[h_2^{(k,l)}\left(\hat{n}_1, \hat{n}_2\right)\right]^3. \tag{67}$$

Inserting Eq. (66) into Eq. (67), we may obtain

$$f_2(\hat{n}_1, \hat{n}_2)g_2\left(\hat{n}_1, \hat{n}_2\right) = \frac{p\left(q^{\hat{n}_1} - 1\right)}{(q-1)^3 \hat{n}_1 \hat{n}_2}\Big[-C_1(q-1)^2 q^{(1-\hat{N})/2}$$
$$+C_2 p(q-1)\left(q^{1-\hat{N}} + q^{1-\hat{n}_2} - 2q^{(1-\hat{N})/2}\right)$$
$$+C_3 p^2\Big(3q^{1-\hat{N}} + 3q^{1-\hat{n}_2} - q^{3(1-\hat{N})/2} - 3q^{(1-\hat{N})/2}$$
$$-q^{(3-\hat{N})/2-\hat{n}_2} - q^{(3+\hat{M})/2-\hat{n}_2}\Big)\Big]. \tag{68}$$

Similarly, we will discuss in the following the unitary realizations and the non-unitary realization of $\mathcal{R}_{q,p}^{c_1,c_2,c_3}$ for the simple case of $(k, l) = (1, 1)$.

(a) The unitary realizations.

The unitary relations $B_2^{\dagger}(J_{\pm}) = B_2(J_{\mp})$ require $f_2(\hat{n}_1, \hat{n}_2) = g_2(\hat{n}_1, \hat{n}_2)$, then substituting the expression of $f_2(\hat{n}_1, \hat{n}_2)$ given by Eq. (68) into Eq. (65), we may obtain

$$\breve{B}_2(J_3) = \frac{p}{q-1}\left(1 - q^{-(1+\hat{N})/2}\right),$$
$$\breve{B}_2(J_+) = \sqrt{\frac{p(q^{\hat{n}_1} - 1)}{(q-1)^3 \hat{n}_1 \hat{n}_2}\Big[-C_1(q-1)^2 q^{(1-\hat{N})/2}}$$
$$\overline{+C_2 p(q-1)\left(q^{1-\hat{N}} + q^{1-\hat{n}_2} - 2q^{(1-\hat{N})/2}\right)}$$
$$\overline{+C_3 p^2\Big(3q^{1-\hat{N}} + 3q^{1-\hat{n}_2} - q^{3(1-\hat{N})/2} - 3q^{(1-\hat{N})/2}}$$
$$\overline{-q^{(3-\hat{N})/2-\hat{n}_2} - q^{(3+\hat{M})/2-\hat{n}_2}\Big)\Big]^{1/2}} a_1^{+} a_2^{+}, \tag{69}$$
$$\breve{B}_2(J_-) = a_1 a_2 \sqrt{\frac{p(q^{\hat{n}_1} - 1)}{(q-1)^3 \hat{n}_1 \hat{n}_2}}\Big[-C_1(q-1)^2 q^{(1-\hat{N})/2}$$

$$+C_2 p(q-1)\left(q^{1-\hat{N}}+q^{1-\hat{n}_2}-2q^{(1-\hat{N})/2}\right)$$
$$+C_3 p^2\left(3q^{1-\hat{N}}+3q^{1-\hat{n}_2}-q^{3(1-\hat{N})/2}-3q^{(1-\hat{N})/2}\right.$$
$$\left.\left.-q^{(3-\hat{N})/2-\hat{n}_2}-q^{(3+\hat{M})/2-\hat{n}_2}\right)\right]^{1/2}.$$

It is easy to check that the realization (69) obeys the commutation relations of $\mathcal{R}_{q,p}^{c_1,c_2,c_3}$.

Inserting Eq. (69) into Eq. (3), the Casimir invariant $\mathcal{C}$ of $\mathcal{R}_{q,p}^{c_1,c_2,c_3}$ reads

$$\mathcal{C}=\frac{\eta(\hat{M}/2)}{(q-1)^3}\left[C_1(q-1)^2+C_2(q-1)\eta(\hat{M}/2)+C_3\eta^2(\hat{M}/2)\right], \tag{70}$$

where $\eta(\hat{M})=p(1-q^{\hat{M}+1/2})$. Calculating the expectation value of $\mathcal{C}$ in the Fock space $\mathcal{F}_2$ gives

$$\langle n_1 n_2|\mathcal{C}|n_1 n_2\rangle=\frac{\eta(j)}{(q-1)^3}\left[C_1(q-1)^2+C_2(q-1)\eta(j)+C_3\eta^2(j)\right], \tag{71}$$

where $j=M/2$, $M=n_1-n_2$ is the eigenvalue of $\hat{M}$.

It is clear from Eqs. (69) and (70) that $\breve{B}_2(J_3)$ depends on the total particle number operator $\hat{N}$ only, while $\mathcal{C}$ the particle number difference operator $\hat{M}$ only, which are different from those for the first kind of double-boson realization given in the last subsection.

Finally, let us end this subsection by giving the *symmetric* results for $\mathcal{R}_{q,p}^{c_1,c_2,c_3}$. First, exchanging $\hat{n}_1$ and $\hat{n}_2$ in Eqs. (69) and (70), then combining them with the respective original one, we may obtain the new double-boson realization

$$\breve{B}_3(J_3)=\frac{p}{q-1}\left(1-q^{-(1+\hat{N})/2}\right),$$
$$\breve{B}_3(J_+)=\sqrt{\frac{1}{2(q-1)^3\hat{n}_1\hat{n}_2}}\left[C_1 p(q-1)^2 q^{(1-\hat{N})/2}\left(-q^{\hat{n}_1}-q^{\hat{n}_2}+2\right)\right.$$
$$+C_2 p^2(q-1)\left(-2q^{1-\hat{N}}+q^{1+\hat{M}}+q^{1-\hat{M}}+4q^{(1-\hat{N})/2}-2q^{(1+\hat{M})/2}\right.$$
$$\left.-2q^{(1-\hat{M})/2}\right)+C_3 p^3\left(-6q^{1-\hat{N}}+3q^{1+\hat{M}}+3q^{1-\hat{M}}\right.$$
$$+2q^{3(1-\hat{N})/2}+6q^{(1-\hat{N})/2}-q^{3(1+\hat{M})/2}$$
$$\left.\left.-3q^{(1+\hat{M})/2}-q^{3(1-\hat{M})/2}-3q^{(1-\hat{M})/2}\right)\right]^{1/2}a_1^+a_2^+, \tag{72}$$
$$\breve{B}_3(J_-)=a_1 a_2\sqrt{\frac{1}{2(q-1)^3\hat{n}_1\hat{n}_2}}\left[C_1 p(q-1)^2 q^{(1-\hat{N})/2}\left(-q^{\hat{n}_1}-q^{\hat{n}_2}+2\right)\right.$$
$$+C_2 p^2(q-1)\left(-2q^{1-\hat{N}}+q^{1+\hat{M}}+q^{1-\hat{M}}+4q^{(1-\hat{N})/2}-2q^{(1+\hat{M})/2}\right.$$
$$\left.-2q^{(1-\hat{M})/2}\right)+C_3 p^3\left(-6q^{1-\hat{N}}+3q^{1+\hat{M}}+3q^{1-\hat{M}}\right.$$

$$+2q^{3(1-\hat{N})/2} + 6q^{(1-\hat{N})/2} - q^{3(1+\hat{M})/2}$$
$$-3q^{(1+\hat{M})/2} - q^{3(1-\hat{M})/2} - 3q^{(1-\hat{M})/2}\Big)\Big]^{1/2},$$

and the new Casimir invariant

$$\mathcal{C} = \frac{1}{(q-1)^3}\left[-C_1(q-1)^2\eta^{(1)} + C_2(q-1)\eta^{(2)} - C_3\eta^{(3)}\right], \tag{73}$$

where $\eta^{(i)} = p[(1-q^{(1+\hat{M})/2})^i + (1-q^{(1-\hat{M})/2})^i]/2$. It is not difficult to check that the realization (72) obeys the commutation relations of $\mathcal{R}_{q,p}^{c_1,c_2,c_3}$, and $\mathcal{C}$ given by Eq. (73) commutes with $\breve{B}_3(J_\mu)$ given by Eq. (72). We observe that indeed there exist explicit symmetries in Eqs. (72) and (73), that is, exchanging $\hat{n}_1$ and $\hat{n}_2$ leaves these results invariant. Although the expression (73) is very different from the expression (58) given in the last subsection, their eigenvalues in the Fock space are the same under some limiting values, for example, $q = p = 1$.

(b) The non-unitary realization.

It follows from Eq. (68) that $g_2(\hat{n}_1, \hat{n}_2) = 1$ may immediately give rise to the non-unitary double-boson realization

$$\begin{aligned}
\bar{B}_2(J_3) &= \frac{p}{q-1}\left(1 - q^{-(1+\hat{N})/2}\right),\\
\bar{B}_2(J_+) &= \frac{p\left(q^{\hat{n}_1}-1\right)}{(q-1)^3\hat{n}_1\hat{n}_2}\Big[-C_1(q-1)^2q^{(1-\hat{N})/2}\\
&\quad +C_2p(q-1)\left(q^{1-\hat{N}} + q^{1-\hat{n}_2} - 2q^{(1-\hat{N})/2}\right)\\
&\quad +C_3p^2(3q^{1-\hat{N}} + 3q^{1-\hat{n}_2} - q^{3(1-\hat{N})/2} - 3q^{(1-\hat{N})/2}\\
&\quad -q^{(3-\hat{N})/2-\hat{n}_2} - q^{(3+\hat{M})/2-\hat{n}_2}\Big)\Big]a_1^+a_2^+,\\
\bar{B}_2(J_-) &= a_1a_2.
\end{aligned} \tag{74}$$

The Casimir invariant $\mathcal{C}$ has the same expression as Eq. (70). Similar to (56), the acting spaces of $\bar{B}_2(J_\mu)$ are the whole Fock space $\mathcal{F}_2$.

The symmetric forms of the non-unitary double-boson realization (74) may also be obtained by using the same method as used in the last subsection, however, they are not given here.

(c) Unitarization of the non-unitary realization.

Now we discuss the connection between the non-unitary realization (74) and the unitary realization (69). Similarly, here we restrict ourselves to the special case $q = p = 1$, i.e. $\bar{B}_2'(J_\mu) = \lim_{q,p=1}\bar{B}_2(J_\mu)$ and $\breve{B}_2'(J_\mu) = \lim_{q,p=1}\breve{B}_2(J_\mu)$.

Denoting the similarity transformation by S_2, we have

$$S_2\bar{B}_2'(J_\mu)S_2^{-1} = \breve{B}_2'(J_\mu), \quad \mu = 3, \pm. \tag{75}$$

Here S_2 depends only on the particle number operators $\hat{n}_1$ and $\hat{n}_2$ too.

Using Eq. (75) and the unitary conditions $(\breve{B}'_2(J_\pm))^\dagger = \breve{B}'_2(J_\mp)$, we may obtain the following unitarization equations

$$U_2^{-1}\left(\bar{B}'_2(J_3)\right)^\dagger U_2 = \bar{B}'_2(J_3),\\ U_2^{-1}\left(\bar{B}'_2(J_\pm)\right)^\dagger U_2 = \bar{B}'_2(J_\mp), \tag{76}$$

where $U_2 \equiv S_2^\dagger S_2$. Calculating the matrix element of Eq. (76) in the Fock space $\mathcal{F}_2$, and using Eq. (74) with $q = p = 1$, we have

$$\left\{4C_1 + 4C_2(n_2-1) + C_3\left[n_1^2 + 3(n_2-1)^2\right]\right\} S_2(n_1, n_2)^2 \\ +4n_2 S_2(n_1-1, n_2-1)^2 = 0. \tag{77}$$

Solving Eq. (77) gives

$$S_2(\hat{n}_1, \hat{n}_2) = \sqrt{\frac{(-1)^{1+\hat{n}_1} C_3^{1-\hat{n}_1}\Gamma(1+\hat{n}_2)\Gamma(\omega_2^+(\hat{M}))\Gamma(\omega_2^-(\hat{M}))}{\Gamma(2-\hat{M})\Gamma(\omega_2^+(\hat{M})+\hat{n}_1-1)\Gamma(\omega_2^-(\hat{M})+\hat{n}_1-1)}}, \tag{78}$$

where

$$\omega_2^\pm(\hat{x}) = \frac{1}{4C_3} \\ \times\left[2C_2 + C_3(5-3\hat{x}) \pm \sqrt{-16C_1C_3 + [2C_2 - C_3(1+\hat{x})][2C_2 + 3C_3(1+\hat{x})]}\right].$$

3.4 The irreducible representation of $\mathcal{R}_{q,p}^{c_1,c_2,c_3}$

Similar to SU(2), $\mathcal{R}_{q,p}^{c_1,c_2,c_3}$ has the Casimir invariant $\mathcal{C}$, hence, making use of the very parallel treatment of angular momentum in quantum mechanics [30], it is not difficult to obtain the unitary representation of $\mathcal{R}_{q,p}^{c_1,c_2,c_3}$ in the common eigenvectors (called the angular momentum basis) $\{|jm\rangle | m = -j, -j+1, \ldots, j\}$ of the elements $\{\mathcal{C}, J_3\}$, with j and m labeling the eigenvalues of $\mathcal{C}$ and J_3 respectively. Here, as an application of boson realizations, we now use one of double-boson realizations to calculate the irreducible representation of $\mathcal{R}_{q,p}^{c_1,c_2,c_3}$.

It is well known that the explicit connection between the particle numbers $\{n_1, n_2\}$ and the angular momentum quantum numbers $\{j, m\}$ is given by [30]

$$j = \frac{1}{2}(n_1 + n_2), \quad m = \frac{1}{2}(n_1 - n_2). \tag{79}$$

We observe from Eqs. (57) and (56) that the same connection exists for the first kind of unitary double-boson realization, i.e., $\mathcal{C}$ depends on the total particle number operator $\hat{N} = \hat{n}_1 + \hat{n}_2$ only and $\breve{B}_1(J_3)$ the particle number difference operator $\hat{M} = \hat{n}_1 - \hat{n}_2$

only. Thus, using the first kind of unitary double-boson realization (56) and Eq. (79), we may obtain

$$\begin{aligned} J_3|jm\rangle &= \frac{p}{q-1}(1-q^{-m})|jm\rangle, \\ J_+|jm\rangle &= \sqrt{\frac{pq^{-m}(q^{j+m+1}-1)}{(q-1)^3}}\Big[-C_1(q-1)^2 + C_2 p(q-1)\left(q^{j+1}+q^{-m}-2\right) \\ &\quad -C_3 p^2\left(q^{2(j+1)}-3q^{j+1}+q^{-2m}-3q^{-m}+q^{j-m+1}+3\right)\Big]^{1/2}|jm+1\rangle, \\ J_-|jm\rangle &= \sqrt{\frac{pq^{1-m}(q^{j+m}-1)}{(q-1)^3}}\Big[-C_1(q-1)^2+C_2 p(q-1)\left(q^{j+1}+q^{1-m}-2\right) \\ &\quad -C_3 p^2\left(q^{2(j+1)}-3q^{j+1}+q^{2(1-m)}-3q^{1-m}+q^{j-m+2}+3\right)\Big]^{1/2}|jm-1\rangle. \end{aligned} \tag{80}$$

When $q = p = C_1 = C_2 = 1$ and $C_3 = 0$, Eq. (80) becomes the standard form of the irreducible representation of SU(2) [30].

4 Conclusion

In this paper we have obtained the explicit expressions for the master representation of $\mathcal{R}_{q,p}^{c_1,c_2,c_3}$ on the space of its universal enveloping algebra $U(\mathcal{R})$ and other indecomposable (irreducible) representations subduced on some invariant subspaces of $U(\mathcal{R})$ or induced on some quotient spaces $U(\mathcal{R})/I_i$ with I_is being the left ideals with respect to $U(\mathcal{R})$. For $U(\mathcal{R})$, we may choose the other bases by arranging the three generators J_+, J_-, J_3 in different sequences, for example, $\{J_3^n J_+^m J_-^r\}$, then the corresponding indecomposable (irreducible) representations can be obtained by the similar approach. They may be related to each other by symmetry considerations. Furthermore, we have obtained various boson realizations of $\mathcal{R}_{q,p}^{c_1,c_2,c_3}$ such as single-boson, single inverse boson, and double-boson (symmetric) realizations by generalizing the usual boson realizations of SU(2) and SU(1,1). It is worth mentioning that in the single-boson realizations, the solution (32) of Eq. (30) is not unique because it allows an extra term $aq^{b-\hat{n}/k}$, where a and b are arbitrary constants. Similar properties exist for the single inverse boson realizations and the double-boson realizations. We have revealed the fact that the nonunitary realizations and the unitary ones may be related by the similarity transformations, which have been obtained by solving the corresponding unitarization equations.

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